ON AN EXACT SOLUTION OF NONSTATIONARY CONVECTION EQUATIONS

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1. Equations of the problem and nature of solutions. It is well known that in the gravitational field $\mathbf{g}$ a nonuniformly heated fluid can be in equilibrium only when the temperature gradient is vertical

$$\mathbf{g} = -\boldsymbol{\gamma} g, \qquad \nabla T_0 = A\boldsymbol{\gamma} \qquad (\boldsymbol{\gamma}^2 = 1) \tag{1.1}$$

If A = const , then the following equations are obtained for small perturbations of equilibrium (they are proportional to $e^{-\lambda t}$)):

$$-\lambda \mathbf{u} = -\nabla p + \nabla^2 \mathbf{u} \pm C\boldsymbol{\gamma} T$$

$$-\lambda PT = \nabla^2 T - C\boldsymbol{\gamma}\mathbf{u}, \quad \operatorname{div} \mathbf{u} = 0 \quad (C^2 = \alpha g R^4 \mid A \mid / \nu\chi, \; P = \nu / \chi) \tag{1.2}$$

Here all quantities are nondimensional; the length R (which characterizes the dimension of the cavity), the time R^2 / ν, the velocity ν / R , the temperature $(\nu / R) (\mid A \mid \nu / \alpha g \chi)^{1/2}$; are selected as units; nondimensional parameters are C^2 the Rayleigh number, and P the Prandtl number. One of the terms in the equations has the ± sign. Here and everywhere in the following presentation the upper sign refers to the case when $A > 0$ (the fluid is heated from above), the lower sign refers to the case when $A < 0$ (heating from below). The system of equations (1.2) has an infinite sequence of solutions for pairs of functions $\{\mathbf{u}_\alpha, T_\alpha\}$ and for decrements λ_α. These solutions are orthogonal to each other in the following sense:

$$\int \{\mathbf{u}_\alpha \mathbf{u}_\beta \mp PT_\alpha T_\beta\} \, dV = C\delta_{\alpha\beta} \qquad (C = \text{const}) \tag{1.3}$$

The perturbation with the index α is monotonous if $\operatorname{Im} \lambda_\alpha = 0$. The perturbation decays if $\operatorname{Re} \lambda_\alpha > 0$.

In a fluid heated from below, perturbations either decay monotonously or grow monotonously [1] so that for $A < 0$ the equilibrium may either be stable or unstable.

When a fluid is heated from above ($A > 0$) all perturbations decay, but not necessarily monotonously [1]. From (1.2) results the following integral relationship:

$$(\lambda - \lambda^*) \int \{\mathbf{u}^* \mathbf{u} - PT^* T\} \, dV = 0 \tag{1.4}$$

from this it is evident that complex λ are possible whenthe integral in (1.4) is equal to zero. For monotonous perturbations this integral coincides with the normalizing integral (1.3). There are two types of monotonous perturbations – "thermal" $\{\mathbf{u}_{1\alpha}, T_{1\alpha}\}$ and "hydrodynamic" $\{\mathbf{u}_{2\alpha}, T_{2\alpha}\}$. For these we have

$$\int \mathbf{u}_{1\alpha}^2 dV < P \int T_{1\alpha}^2 dV, \qquad \int \mathbf{u}_{1\alpha}^2 dV > P \int T_{2\alpha}^2 dV$$

normalizing integrals of different types of perturbations have therefore different signs. For $C \to 0$ in "thermal" perturbations the velocity disappears and only the temperature remains, in "hydrodynamic" perturbations the opposite is the case.

An entirely analogous situation was once before encountered by one of the authors in the investigation [2] of the spectrum of perturbations in a conducting fluid in a magnetic field. Heating from above makes the equations not self-conjugate and so similar to equations in magnetohydrodynamics that the following assertions can be made [2]. In a fluid heated from above there are no oscillatory perturbations for small values of C. From some $C = C_*$, decrements of two monotonous perturbations of different type and identical symmetry can intersect. Then for $C > C_*$, instead of two monotonous perturbation in the spectrum, two oscillatory perturbations appear with complex conjugate decrements. At the very point C_* one of the normalizing integrals becomes zero. These theoretical conclusions are fully confirmed by the example given below. For this example Equation (1.2) has an exact solution.

2. The case of the exact solution. Let us examine perturbations of equilibrium in a fluid in a spherical cavity heated from above or from below. In spherical coordinates r, ϑ, φ with the polar axis along γ and with the following boundary conditions

$$\mathbf{u} = 0, \quad T = 0 \quad \text{for } r = 1; \qquad \mathbf{u},\ T \text{ bounded for } r = 0 \tag{2.1}$$

Equations (1.2) have a class of exact solutions with the following structure (*)

$$\mathbf{u} = v(r)\, \mathbf{r} \times \nabla\, [\sin m\varphi P_l^m (\cos\vartheta)]$$

$$T = \theta(r) \cos m\varphi\, P_l^m (\cos\vartheta), \qquad p = f(r,\vartheta) \cos m\varphi \tag{2.2}$$

Here $P_l^m(\cos\vartheta)$ are associated Legendre polynomials. Substitution of (2.2) into Equations (1.2) yields for example

$$\nabla^2 \mathbf{u} = \left[\varphi_1 \sin m\varphi P_l^m (\cos\vartheta) - \vartheta_1 \frac{m \cos m\varphi}{\sin\vartheta} P_l^m (\cos\vartheta)\right] L v(r)$$

$$\left(L = \frac{d^2}{dr^2} + 2\frac{d}{dr} - \frac{l(l+1)}{r^2}\right)$$

$$\gamma\mathbf{u} = (\mathbf{r}_1 \cos\vartheta - \vartheta_1 \sin\vartheta)\, v(r) \left[\varphi_1 \sin m\varphi\, (P_l^m)^{\cdot} - \vartheta_1 \frac{m\cos m\varphi}{\sin\vartheta} P_l^m\right] =$$

$$= m v(r) \cos m\varphi P_l^m (\cos\vartheta)$$

The dot indicates here differentiation with respect to ϑ.

Scalar multiplication of the first equation of the system (1.2) in turn by ϑ_1 and φ_1, yields

$$(\lambda + L)\, v(r) \frac{m}{\sin\vartheta} P_l^m (\cos\vartheta) = -\frac{f^{\cdot}(r,\vartheta]}{r} \mp C\theta(r) \sin\vartheta\, P_l^m (\cos\theta) \tag{2.3}$$

$$(\lambda + L)\, \bar{v}(r)\, P_l^{m\cdot} (\cos\vartheta) = -\frac{m}{r\sin\vartheta} f(r,\vartheta) \tag{2.4}$$

Calculating from the last equation the derivative

$$-f^{\cdot}(r,\vartheta) = (\lambda + L)\frac{rv(r)}{m} [\sin\vartheta\, (P_l^m(\cos\vartheta))^{\cdot\cdot} + \cos\vartheta\, (P_l^m(\cos\vartheta))^{\cdot}] =$$

$$= (\lambda + L)\frac{rv(r)}{m} \sin\vartheta \left[\frac{m^2}{\sin^2\theta} - l(l+1)\right] P_l^m (\cos\vartheta)$$

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*) These solutions were found by Sorokin (see note with respect to reference [3]).

and substituting it into (2.3) we obtain after simplification

$$\frac{l(l+1)}{m}(\lambda + L)\, v(r) = \mp C\theta(r) \tag{2.5}$$

The second equation of the system (1.2) gives

$$(\lambda P + L)\, \theta(r) = mCv(r) \tag{2.6}$$

Thus, for finding radial functions $v(r)$ and $\theta(r)$ we have the system of equations

$$\begin{gathered} v'' + \frac{2v'}{r} + \left[\lambda - \frac{l(l+1)}{r^2}\right] v = \mp \frac{mC}{l(l+1)}\theta \\ \theta'' + \frac{2\theta'}{r} + \left[\lambda - \frac{l(l+1)}{r^2}\right]\theta = mCv \end{gathered} \tag{2.7}$$

This system of equations must be solved with the following boundary conditions

$$v(1) = 0, \qquad \theta(1) = 0; \qquad v(0),\ \theta(0) - \text{finite} \tag{2.8}$$

We will look for a specific solution of (2.7) and (2.8) in the form

$$v(r) = \frac{B}{\sqrt{r}} J_{l+1/2}(kr), \qquad \theta(r) = \frac{D}{\sqrt{r}} J_{l+1/2}(kr) \tag{2.9}$$

Here $J_{l+1/2}(kr)$ is a Bessel function of the first kind. For B and D we obtain two algebraic equations

$$(\lambda - k^2) B = \mp \frac{mC}{l(l+1)} D, \qquad (\lambda P - k^2) D = mCB \tag{2.10}$$

which are consisten if

$$(\lambda - k^2)(\lambda P - k^2) \pm \frac{m^2C^2}{l(l+1)} = 0 \tag{2.11}$$

From this two values, k_1^2 and k_2^2, are obtained for k^2 so that the general solution, bounded at the origin of coordinates, for the system of equations (2.7), has the form

$$v(r) = \frac{B_1}{\sqrt{r}} J_{l+1/2}(k_1 r) + \frac{B_2}{\sqrt{r}} J_{l+1/2}(k_2 r), \qquad \theta(r) = \frac{D_1}{\sqrt{r}} J_{l+1/2}(k_1 r) + \frac{D_2}{\sqrt{r}} J_{l+1/2}(k_2 r) \tag{2.12}$$

Among the four constants entering into (2.12) only two will be independent; according to (2.10)

$$B_1 = \mp \frac{mCD_1}{l(l+1)(\lambda - k_1^2)}, \qquad D_2 = \frac{mCB_2}{(\lambda P - k_2^2)} \tag{2.13}$$

Coefficients D_1 and B_2 are subject to determination from boundary conditions (2.8) on the surface of the fluid sphere. It is easy to see that two types of solutions exist which satisfy these conditions

$$1)\ B_2 = 0, \quad \theta_1(r) = \frac{D_1}{\sqrt{r}} J_{l+1/2}(k_1 r), \qquad v_1(r) = \mp \frac{mCD_1}{l(l+1)(\lambda - k_1^2)} \frac{J_{l+1/2}(k_1 r)}{\sqrt{r}} \tag{2.14}$$

$$2)\ D_1 = 0, \qquad v_2(r) = \frac{B_2}{\sqrt{r}} J_{l+1/2}(k_2 r), \qquad \theta_2(r) = \frac{mCB_2}{\lambda P - k_2^2} \frac{J_{l+1/2}(k_2 r)}{\sqrt{r}} \tag{2.15}$$

According to the clasification adopted in Section 1, the solution (2.14) corresponds to "thermal" and (2.15) to "hydrodynamic perturbations . For both pertutbations, the conditions (2.8) yield

$$J_{l+1/2}(k_n) = 0 \quad (n = 1, 2, 3, \ldots) \tag{2.16}$$

Here n is the number of nodes of the radial function. Equations (2.11) and (2.16) determine decrements λ_1 of "thermal" and decrements λ_2 of "hydrodynamic" perturbations as functions of Rayleigh number C^2

$$\begin{aligned}\lambda_{1,\,lmn} &= \frac{1}{2P}\left[(P+1)\,k_n^2 - \left((P-1)^2\,k_n^4 \mp \frac{4m^2C^2P}{l\,(l+1)}\right)^{1/2}\right]\\ \lambda_{2,\,lmn} &= \frac{1}{2P}\left[(P+l)\,k_n^2 + \left((P-1)^2\,k_n^4 \mp \frac{4m^2C^2P}{l\,(l+1)}\right)^{1/2}\right]\end{aligned} \qquad (2.17)$$

We also present the value for the normalizing integral, for example, for the "hydrodynamic" perturbations

$$\int \{\mathbf{u}^2_{1,\,lmn} \mp PT^2_{1,\,lmn}\}\,dV = J^2_{l-1/2}(k_n)\,\frac{\pi}{2l+1}\,\frac{(l+m)!}{(l-m)!}\left[l\,(l+1) \mp \frac{Pm^2C^2}{(\lambda P - k_1^2)^2}\right] \qquad (2.18)$$

The critical value C_0 above which the equilibrium of a fluid heated from below is unstable with respect to a perturbation with specific l, m, n is found from the condition

$$\lambda_{lmn} = 0$$

Computations yield

$$C_0^2 = \frac{l\,(l+1)}{m^2}\,k_n \qquad (2.19)$$

In heating from above, critical C_*, can be reached for which λ_1 and λ_2 as determined from (2.17) coincide

$$\lambda_1 = \lambda_2 = \lambda_* = \frac{P+1}{2P}\,k_n^2$$

$$C_*^2 = \frac{(P-1)^2 k_n^4}{4Pm^2}\,l\,(l+1) \qquad (2.20)$$

while integral (2.18) becomes zero. For $C > C_*$ two oscillatory perturbations appear oscillating with the frequency

$$\left[\frac{m}{Pl\,(l+1)}\right]^{1/2}\sqrt{C^2 - C_*^2} \qquad (2.21)$$

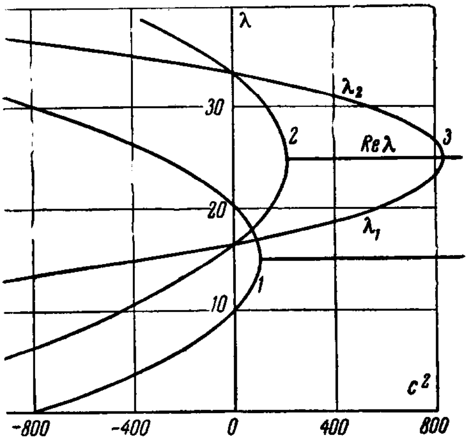

Fig. 1

Real parts of decrements of these pertutbations do not depend on C and are equal to λ_* .

We note that the relationship

$$\frac{C_*^2}{C_0^2} = \frac{(P-1)^2}{4P} \qquad (2.22)$$

does not depend on the index of perturbation and will be general for the entire spectrum of decrements. Several lower decrements for a fluid with $P = 2$ are presented in Fig.1 where curve 1 corresponds to values $l = m = n = 1$, curve 2 to values $l = m = 2$, $n = 1$ and curve 3 to values $l = 2$, $m = n = 1$. Heating from below corresponds to negative Rayleigh numbers.

The problem which was examined can also be solved for other boundary conditions. For example, in the case of thermally insulated walls $(\theta'(r) = 0)$ decrements $\lambda(C)$ are found from Equation

$$\frac{J_{l+1/2}(k_1)\,[(l+1)\,J_{l+1/2}(k_2) + k_2 J_{l-1/2}(k_2)]}{J_{l+1/2}(k_2)\,[(l+1)\,J_{l+1/2}(k_1) + k_1 J_{l-1/2}(k_1)]} = \frac{\lambda - k_1^2}{\lambda - k_2^2} \qquad (2.23)$$

where k_1 and k_2 are expressed through λ and C from (2.11).

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